Based on K. H. Rosen: Discrete Mathematics and its Applications.

Lecture 3: Propositional equivalences. Section 1.3

1 Propositional equivalences

Definition 1. A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**. A compound proposition that is always false is called a **contradiction**. A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Example 2. No matter what is the truth value for p, the expression $p \vee \neg p$ is always true, hence a tautology. On the other hand, $p \wedge \neg p$ is never true and therefore is a contradiction.

Definition 3. The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Example 4. For example $\neg\neg p \equiv p$.

Theorem 5. *The De Morgan's laws for conjunctions and disjunctions are:*

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad \textit{and} \qquad \neg(p \vee q) \equiv \neg p \wedge \neg q.$$

Theorem 6. *The distributive laws for any propositions p, q and r states that*

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \qquad \textit{and} \qquad p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r).$$

Example 7. Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert."
Let p be "Miguel has a cellphone" and q be "Miguel has a laptop computer." Then "Miguel has a cellphone and he has a laptop computer" can be represented by $p \wedge q$. By the first of De Morgan's laws, $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$. Consequently, we can express the negation of our original statement as "Miguel does not have a cellphone or he does not have a laptop computer."
Let r be "Heather will go to the concert" and s be "Steve will go to the concert." Then "Heather will go to the concert or Steve will go to the concert" can be represented by $r \vee s$. By the second of De Morgan's laws, $\neg(r \vee s)$ is equivalent to $\neg r \wedge \neg s$. Consequently, we can express the negation of our original statement as "Heather will not go to the concert and Steve will not go to the concert."

In general we have the following logical equivalences

1. (Identity laws)
   (a) $p \vee F \equiv p$
   (b) $p \wedge T \equiv p$
2. (distributive laws)
   (a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
   (b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
3. (Associative laws)
   (a) $p \vee (q \vee r) \equiv (p \vee q) \vee r$
   (b) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
4. (De Morgan laws)
   (a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
   (b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$
5. (Idempotent laws)
   (a) $p \vee p \equiv p$
   (b) $p \wedge p \equiv p$
6. (Domination laws)
   (a) $p \vee T \equiv T$
   (b) $p \wedge F \equiv F$
7. (Double negation law) $\neg(\neg p) \equiv p$
8. (Negation laws)
   (a) $p \vee \neg p \equiv T$
   (b) $p \wedge \neg p \equiv F$
9. (Absortion laws)
   (a) $p \vee (p \wedge q) \equiv p$
   (b) $p \wedge (p \vee q) \equiv p$
10. (Commutative laws)
    (a) $p \wedge q \equiv q \wedge p$
    (b) $p \vee q \equiv q \vee p$

Example 8. Show that $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences. We do the sequence of logical equivalences

$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{(De Morgan)}\\
&\equiv \neg p \wedge \neg(\neg p) \vee \neg q && \text{(De Morgan)}\\
&\equiv \neg p \wedge p \vee \neg q && \text{(double negation)}\\
&\equiv \neg p \wedge p \vee \neg p \wedge \neg q && \text{(distributive)}\\
&\equiv F \vee \neg p \wedge \neg q && \text{(negation)}\\
&\equiv \neg p \wedge \neg q \vee F && \text{(commutative)}\\
&\equiv \neg p \wedge \neg q && \text{(identity)}
\end{aligned}$$

Questions:

(1) Show that $(p \wedge q) \to (p \vee q)$ is a tautology.